

## Stochastic textures and multifractality

- ▶ Visual textures can range from **regular patterns** to complex stochastic structures.
- ▶ While regular textures have identifiable repeated motifs, stochastic textures have no clear geometry: they look noise-like, but their organization is encoded in **statistical dependencies across scales**.

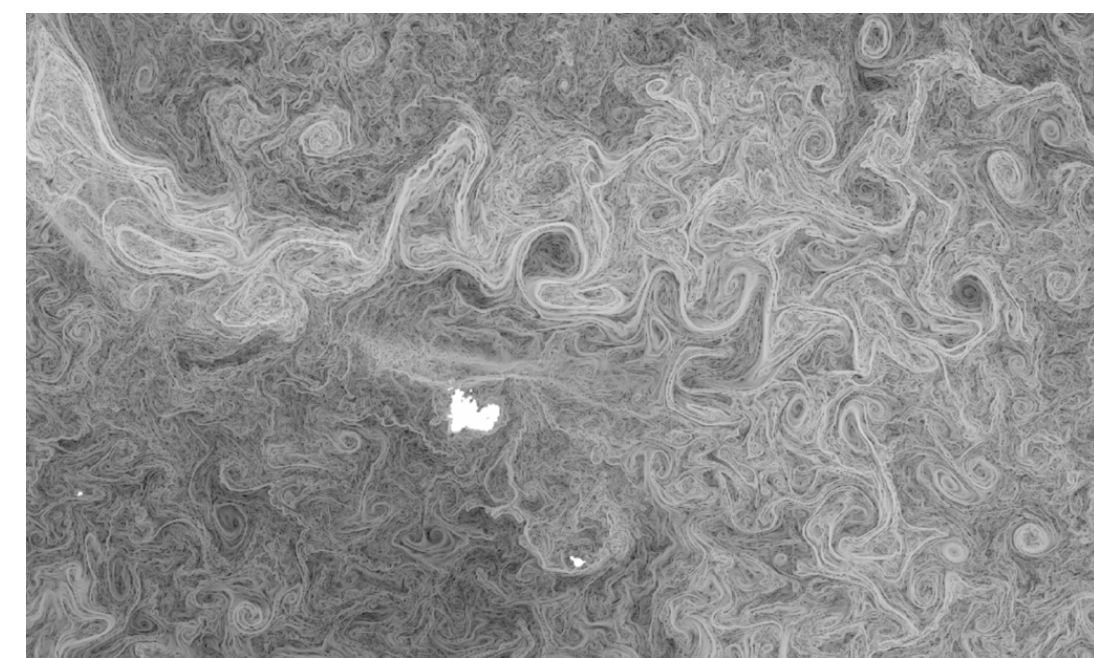
### Regular textures

- ▶ Repeated structured patterns
- ▶ **Periodic** or quasi-periodic organization
- ▶ Visible primitives and geometry



### Stochastic / multifractal textures

- ▶ Irregular, noise-like visual appearance
- ▶ No explicit object or shape
- ▶ Structure encoded by **multiscale statistics**



**Goal.** We study **deep generative models** for **synthesis** and **super-resolution** of multivariate multifractal random walk textures, with quality assessed through **wavelet-based multiscale statistics** rather than visual inspection alone.

## Multifractal Random Walks (MRW)

- ▶ **Multifractal Random Walks (MRW)** [1, 2] model textures with **scale-free statistics** and **spatial intermittency** [3]. They are defined as:

$$X(r) = G_H(r) e^{\omega_\lambda(r)}, \quad (1)$$

where  $G_H(r)$  is a 2D fractional Gaussian noise controlled by the **Hurst exponent**  $H$ , and  $\omega_\lambda(r)$  is a Gaussian modulation field with covariance

$$\mathbb{E}\{\omega_\lambda(r)\omega_\lambda(r')\} = \lambda^2 \log\left(\frac{L}{\|r - r'\| + 1}\right), \quad (2)$$

for  $\|r - r'\| \leq L$ . The parameter  $\lambda$  controls the strength of **multifractality**.

- ▶ **Multivariate extension**: several MRW components are jointly generated,

$$X(r) = (X^{(1)}(r), \dots, X^{(M)}(r)), \quad (3)$$

with inter-channel dependencies controlled by:

- ▶  $\rho$ : second-order correlation between components;
- ▶  $\rho^{mf}$ : cross-multifractal dependency, beyond covariance.

- ▶ **Analysis**: Wavelet coefficients  $d_X(j, k)$  and wavelet leaders  $L_X(j, k)$  yield cumulants:

$$C_1(2^j) = c_1^{(0)} + c_1 \ln 2^j, \quad C_2(2^j) = c_2^{(0)} + c_2 \ln 2^j, \quad (4)$$

$$\rho_{nm} = \mathbb{E}\{G_{H_n}(r)G_{H_m}(r)\} / \lambda_n \lambda_m, \quad \rho_{nm}^{mf} = \mathbb{E}\{\omega_{\lambda_n}(r)\omega_{\lambda_m}(r)\} / \lambda_n \lambda_m, \quad (5)$$

with  $c_1 = H - \lambda^2/2$  and  $c_2 = -\lambda^2$ . The coefficient  $c_1$  mainly reflects correlation structure, while  $c_2$  quantifies multifractality.

- ▶ In this work,  $c_1$ ,  $c_2$ ,  $\rho$ , and  $\rho^{mf}$  are used only as **validation criteria** to assess whether generated and super-resolved textures preserve the target **multiscale statistics**.

## Generative Adversarial Networks (GANs)

**Definition.** A GAN consists of two neural networks trained in competition:

- ▶ the **Generator**  $G$  produces synthetic textures from random noise  $z$ ;
- ▶ the **Discriminator/Critic**  $D$  or  $C$  distinguishes real textures from generated ones.

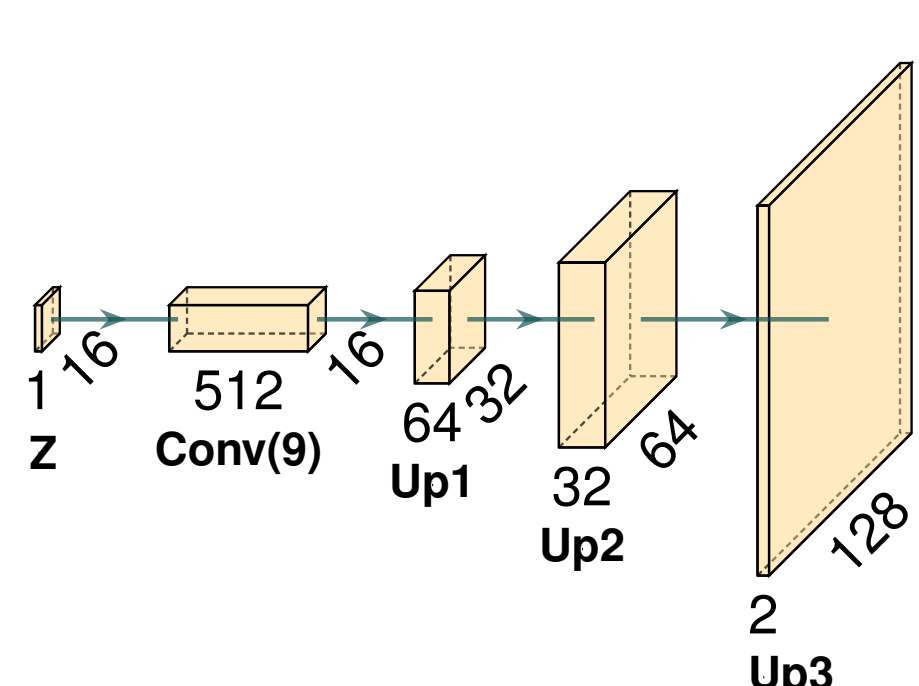
**Loss function.** For a standard DC-GAN [4]:

$$\min_G \max_D \mathbb{E}_{x \sim p_{\text{data}}} [\log D(x)] + \mathbb{E}_{z \sim p_z} [\log(1 - D(G(z)))]. \quad (6)$$

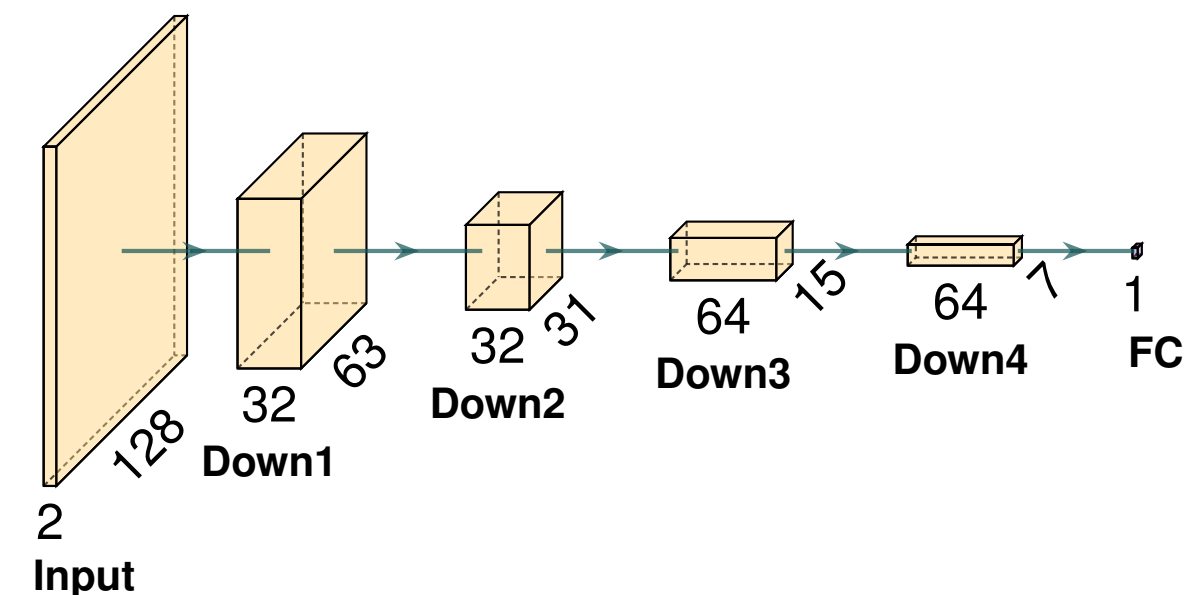
For a Wasserstein GAN we have replaced the Fully Connected Layer with a convolution [5]:

$$\mathcal{L}(C) = \mathbb{E}_{x \sim p_{\text{data}}} [C(x)] - \mathbb{E}_{z \sim p_m} [C(G(z))] + \gamma \mathbb{E}_{y \sim P_*} \left[ \left( \max\{0, \|\nabla_y C(y)\|_2 - 1\} \right)^2 \right].$$

where  $C$  is the critic.



Generator: a single convolution (kernel 9, padding 4) followed by three upsampling + convolution blocks.



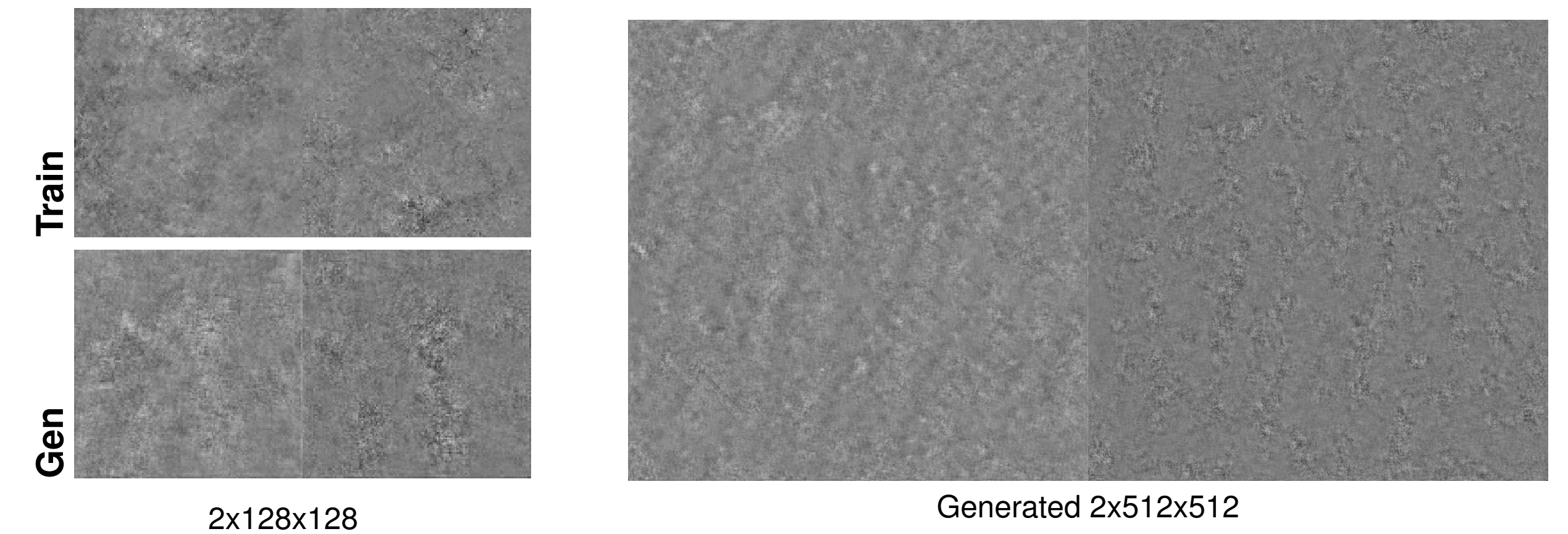
Critic: four down-sampling stages (kernel 3, stride 2 convolutions) with LeakyReLU, batch normalisation, and dropout.

## Training Configuration

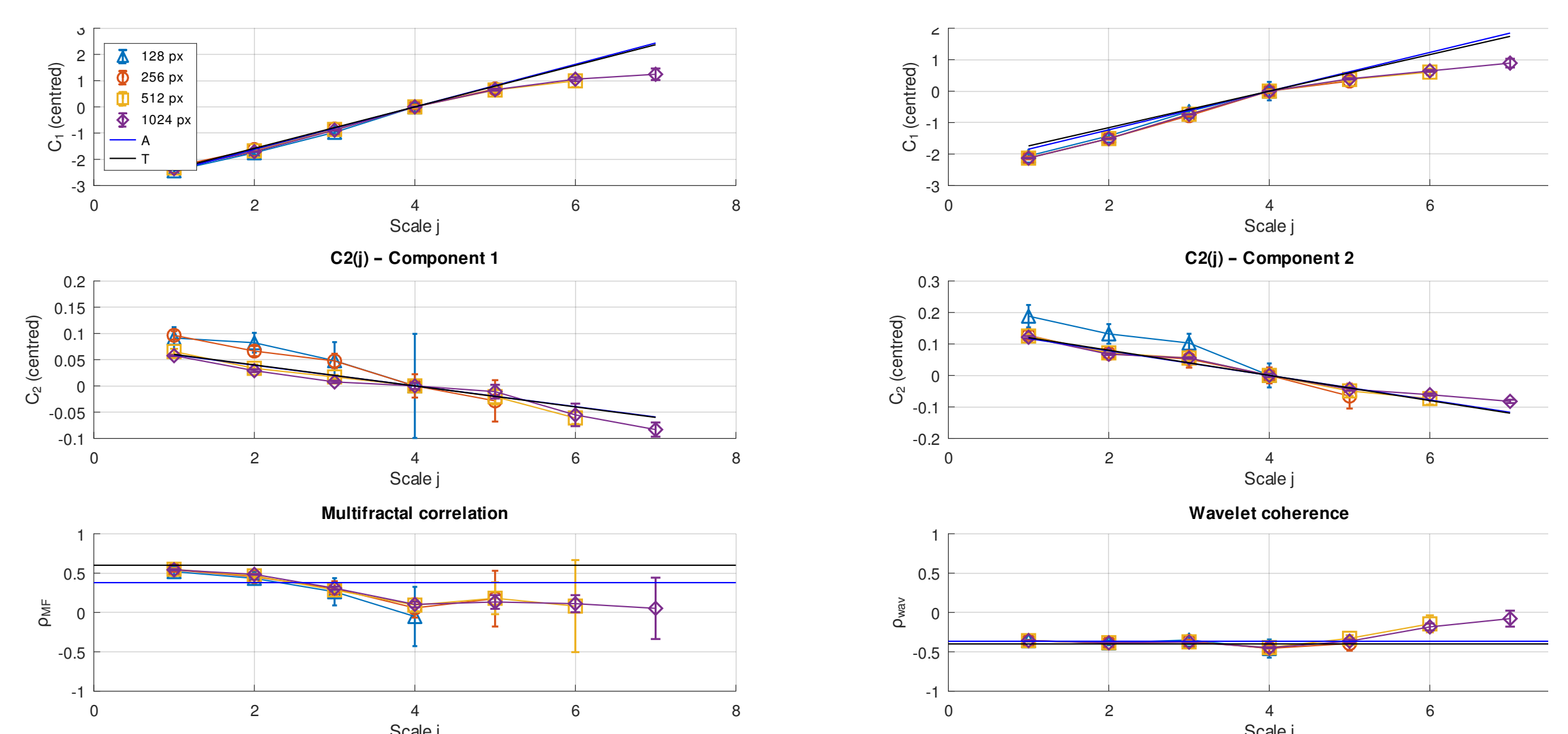
- ▶ The dataset consisted of 8 000 textures of size 2x128x128, synthesized with parameters  $H = [0.8 \ 0.6]$ ,  $\lambda^2 = [0.02 \ 0.04]$ ,  $\rho = -0.4$  and  $\rho_{MF} = 0.6$ .
- ▶ The optimizer used was ADAM ( $\beta_2 = 0.9$ ,  $lr = 10^{-4}$ ), batch size 32, the training updates the Critic 5 times, then updates the Generator. At each epoch we generate 20, 15, 10, 5 textures of resolution 128, 256, 512, 1024 respectively.

## Results

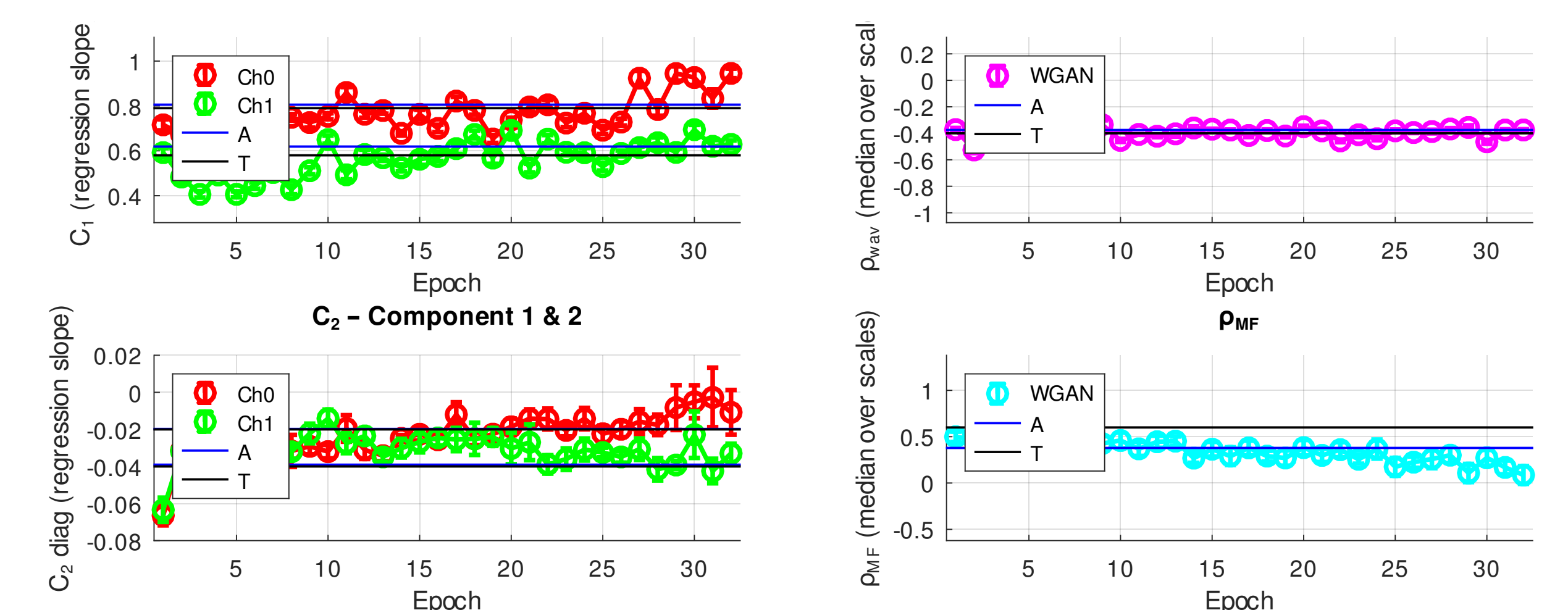
At epoch 28 the generator was able to output these textures, also the reference training texture used.



Statistics over different resolution generated textures. (Approximate lines are over 16 textures 2x512x512)



Performance of statistics per epoch at resolution 2x512x512:



## Conclusion and Future Work

- ▶ **Preliminary results** show that this GAN architecture modification allows the generator to take noise inputs of different sizes, enabling the generation of textures with **higher spatial resolution**.
- ▶ The generated textures reproduce the expected **scaling behavior**, **multifractality**, and **cross-multifractality** with **satisfactory performance** for the first scales.
- ▶ Texture quality was assessed using wavelet-leader statistics  $C_1(j)$ ,  $C_2(j)$ ,  $\rho(j)$ , and  $\rho^{mf}(j)$ , **without embedding these quantities in the loss function**.

**Future work:** testing the approach on **larger multivariate images** and comparing it with **diffusion models** in order to identify the strengths and limitations of each generative framework.

## Acknowledgements and References

We acknowledge the support of the Centre Blaise Pascal's IT test platform at ENS de Lyon (Lyon, France) for Machine Learning facilities.

- [1] B. B. Mandelbrot. "Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier". In: *J. Fluid Mech.* 62 (1974), pp. 331–358.
- [2] E. Bacry, J. Delour, and J.-F. Muzy. "Multifractal random walk". In: *Phys. Rev. E* 64: 026103 (2001).
- [3] S. Jaffard, P. Abry, and H. Wendt. "Irregularities and Scaling in Signal and Image Processing: Multifractal Analysis". In: *Benoit Mandelbrot: A Life in Many Dimensions*. Ed. by Michael Frame and Nathan Cohen. Singapore: World scientific publishing, 2015, pp. 31–116. ISBN: 9789814366069.
- [4] P. Abry, V. Mauduit, E. Quemener, et al. "Multivariate multifractal texture DCGAN synthesis: How well does it work? How does one know?" In: *J. Sign. Process. Syst.* 94 (2022), pp. 179–195. DOI: 10.1007/s11265-021-01701-y.
- [5] Th. Beroud, P. Abry, Y. Malevergne, M. Senneret, G. Perrin, and J. Macq. "Wasserstein GAN Synthesis for Time Series with Complex Temporal Dynamics: Frugal Architectures and Arbitrary Sample-Size Generation". In: *ICASSP 2023 - 2023 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. 2023, pp. 1–5.